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THE THEORY OF QUASI-SASAKIAN STRUCTURES

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Introduction

On a contact manifold of dimension $2n+1$ there exists, by definition, a global 1-form η such that $\eta \wedge (d\eta)^n \neq 0$. An almost contact manifold also carries a 1-form η but it is not necessarily of maximal rank. The purpose of this paper is to explore the meaning of the rank of η. To this end, we initiate the study of normal almost contact metric manifolds with closed fundamental 2-form Φ. Such manifolds will be called quasi-Sasakian manifolds.

§1 presents the basic definitions and some results from the theory of almost contact structures. Beginning with §2 we develop the theory of quasi-Sasakian structures. In §2 a large class of examples is given and in §3 we discuss the meaning of the rank of η. The result is that if η has rank $2p+1$ and the determined almost product structure is integrable then the manifold is locally the product of a Sasakian (normal contact metric) manifold and a Kaehler manifold. That is to say, η having rank $2p+1$ means, loosely speaking, that the space is split locally into a Sasakian piece where $\eta \wedge (d\eta)^p \neq 0$ and a Kaehler piece whose fundamental 2-form is $\Phi - d\eta$ properly restricted. §4 gives some geometric results on quasi-Sasakian manifolds and §5 characterizes the case where $d\eta = 0$, the latter characterization being necessary in the study of the topology of cosymplectic manifolds [1], [2].

1. Almost contact manifolds

All manifolds considered will be C^∞ and connected. A superscript will denote the dimension of the manifold, for example M^{2n+1}, and $\mathscr{E}^{2n+1}$ will denote the module of vector fields over M^{2n+1}. When we speak of an almost contact manifold, quasi-Sasakian manifold, etc., we mean the manifold together with the corresponding structure.

A $(2n+1)$-dimensional manifold carrying a global 1-form η such that $\eta \wedge (d\eta)^n \neq 0$ is said to have a *contact structure* with η as its *contact form*. On the other hand, a manifold M^{2n+1} has an *almost contact structure* (ϕ, ξ, η)

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if it carries a tensor field ϕ of type $(1, 1)$, a vector field ξ, and a 1-form η such that

$$\eta(\xi) = 1, \qquad \phi\xi = 0,$$
$$\eta \circ \phi = 0, \qquad \phi^2 = -I + \xi \otimes \eta; \tag{1.1}$$

this is equivalent to a reduction of the structural group of the tangent bundle of M^{2n+1} to $U(n) \times 1$ (see [9]). From equations (1.1) we see that the maps $-\phi^2$ and $\xi \otimes \eta$ form an almost product structure on M^{2n+1} with decomposition $\mathscr{E}^{2n+1} = \mathscr{E}^{2n} \oplus \mathscr{E}^1$.

Furthermore, an almost contact manifold M^{2n+1} admitting a Riemannian metric g such that

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$$
$$g(X, \xi) = \eta(X) \tag{1.2}$$

where $X, Y \in \mathscr{E}^{2n+1}$, is said to have an *almost contact metric structure* (ϕ, ξ, η, g). It follows from (1.1) that

$$g(\phi X, Y) = -g(X, \phi Y)$$

that is in an almost contact metric manifold with structure tensors (ϕ, ξ, η, g), ϕ is skew-symmetric with respect to g. We define a 2-form Φ by

$$\Phi(X, Y) = g(X, \phi Y)$$

and call it the *fundamental* 2*-form* of the almost contact metric structure. If M^{2n+1} has a contact structure with contact form η then it has an underlying almost contact metric structure (ϕ, ξ, η, g) such that

$$\Phi = d\eta$$

called an *associated almost contact metric structure* [9].

Let M^{2n+1} be an almost contact manifold. S. Sasaki and Y. Hatakeyama [10] defined an almost complex structure J on $M^{2n+1} \times R^1$ by

$$J\left(X, f\frac{d}{dt}\right) = \left(\phi X - f\xi, \eta(X)\frac{d}{dt}\right)$$

where f is a C^∞ real-valued function on $M^{2n+1}R^1$ and $X \in \mathscr{E}^{2n+1}$. Considering the Nijenhuis torsion $[J, J]$ of J, they computed $[J, J]((X, 0), (Y, 0))$ and $[J, J]((X, 0), (0, d/dt))$ which gave rise to four tensors $N^{(1)}, N^{(2)}, N^{(3)}, N^{(4)}$ given by

$$(1.3)\qquad \begin{aligned} N^{(1)}(X, Y) &= [\phi, \phi](X, Y) + d\eta(X, Y)\xi \\ N^{(2)}(X, Y) &= (\mathscr{L}_{\phi X}\eta)(Y) - (\mathscr{L}_{\phi Y}\eta)(X) \\ N^{(3)}(X) &= (\mathscr{L}_{\xi}\phi)X \\ N^{(4)}(X) &= -(\mathscr{L}_{\xi}\eta)(X) \end{aligned}$$

where $\mathscr{L}_X$ denotes the Lie derivative with respect to X. The result is that J is integrable if and only if $N^{(1)} = 0$; in particular, $N^{(1)} = 0$ implies $N^{(2)} = N^{(3)} = N^{(4)} = 0$ [10]. An almost contact structure is said to be *normal* if $N^{(1)} = 0$, that is, if the almost complex structure on $M^{2n+1} \times R^1$ is integrable. A normal contact metric structure is called a *Sasakian structure*.

We now state some results from the theory of almost contact structures which are required later. The first two are due to A. Morimoto [7], [8].

Proposition 1.1. *Suppose that M^{2n+1} is the bundle space of a principal circle bundle over a complex manifold M^{2n} and that there exist a connexion form η on M^{2n+1} and a 2-form Ω, the curvature form of η, of bidegree* (1, 1) *on M^{2n} such that $d\eta = \pi^*\Omega$, where $\pi: M^{2n+1} \to M^{2n}$ is the bundle projection map. Then we can find a linear transformation field ϕ and a vector field ξ on M^{2n+1} such that (ϕ, ξ, η) is a normal almost contact structure.*

For later use we give the definitions of ϕ and ξ in this proposition. Let J be the almost complex structure on M^{2n}, that is $J^2 = -I$. Then ϕ is given by $\phi X = \tilde{\pi} J \pi_* X$ where $\tilde{\pi}$ denotes the horizontal lift with respect to the connexion given by η. The vector field ξ is defined by requiring that it be vertical (i.e., $\pi_*\xi = 0$) and that $\eta(\xi) = 1$.

We say that the vector field ξ on an almost contact manifold M^{2n+1} is *regular* if for every point $m \in M^{2n+1}$ there is a (coordinate) neighborhood U_m of m such that every orbit of ξ passes through U_m at most once. If the orbits of ξ are closed curves, ξ is called a (regular) *closed vector field.*

Morimoto [8] showed that if ξ is a regular closed vector field, then the only normal almost contact manifolds are those constructed above.

Proposition 1.2. *If M^{2n+1} is a normal almost contact manifold with ξ a regular closed vector field, then M^{2n+1} has a principal circle bundle structure over a complex manifold M^{2n} as described in Proposition* 1.1.

As a corollary we have that if M^{2n+1} is a compact regular normal almost contact manifold, then it has a circle bundle structure over a complex manifold as in Proposition 1.1.

The almost complex structure tensor J in this theorem is given by $JX = \pi_*\phi\tilde{\pi}X$. The operator J is well-defined; for, if $\bar{X}$ is a vector field on M^{2n+1} then $\phi\bar{X}$ is a horizontal vector field with respect to the connexion determined by η. Thus, ϕ is invariant under the right translations of M^{2n+1} by the action of S^1 and hence $JX(\pi(m))$ is independent of the choice of m on the fibre over $\pi(m)$. In the proof of Theorem 2.4 below we will show that $J^2 = -I$.

If the manifold M^{2n} in Proposition 1.1 is only an almost complex manifold, then ϕ and ξ as given, together with the connexion form η define an almost contact structure on the bundle space M^{2n+1}. Hatakeyama [4] proved the following proposition.

Proposition 1.3. *The almost contact structure on the circle bundle M^{2n+1} given in Proposition 1.1 is normal if and only if the almost complex structure on the base manifold M^{2n} is integrable and the curvature form Ω of the connexion form η is of bidegree* (1, 1).

2. Quasi-Sasakian structures

Definition. An almost contact metric structure (ϕ, ξ, η, g) is called *quasi-Sasakian* if it is normal and its fundamental 2-form Φ is closed, that is, for every $X, Y \in \mathscr{E}^{2n+1}$

$$(2.1)\qquad \begin{aligned} &[\phi, \phi](X, Y) + d\eta(X, Y)\xi = 0\,, \\ &d\Phi = 0\,,\ \Phi(X, Y) = g(X, \phi Y)\,. \end{aligned}$$

There are many types of quasi-Sasakian structures ranging from the cosymplectic case, $d\eta = 0$ (rank $\eta = 1$), to the Sasakian case, $\eta \wedge (d\eta)^n \neq 0$ (rank $\eta = 2n + 1$, $\Phi = d\eta$). The 1-form η has *rank* $r = 2p$ if $(d\eta)^p \neq 0$ and $\eta \wedge (d\eta)^p = 0$, and has *rank* $r = 2p + 1$ if $\eta \wedge (d\eta)^p \neq 0$ and $(d\eta)^{p+1} = 0$. We also say that r is the *rank* of the quasi-Sasakian structure.

We shall first show that there are no quasi-Sasakian structures of even rank.

Lemma 2.1. *If (ϕ, ξ, η, g) is a normal almost contact metric structure, then*

$$d\eta(X, \xi) = 0$$

for every $X \in \mathscr{E}^{2n+1}$.

Proof. The coboundary formula for d gives

$$\begin{aligned} d\eta(X, \xi) &= X(\eta(\xi)) - \xi(\eta(X)) - \eta([X, \xi]) \\ &= -\,\xi(\eta(X)) - \eta([X, \xi]) \\ &= -\,(\mathscr{L}_\xi \eta)(X) = 0 \end{aligned}$$

since $\eta(\xi) = 1$ and by normality (see formula (1.3)), $(\mathscr{L}_\xi\eta)(X) = 0$.

Theorem 2.2. *There are no quasi-Sasakian structures of even rank.*

Proof. Let $X_1, \cdots, X_{2p} \in \mathscr{E}^{2n+1} = \mathscr{E}^{2n} \oplus \mathscr{E}^1$ be vector fields such that $(d\eta)^p(X_1, \cdots, X_{2p}) \neq 0$. By Lemma 2.1 we may assume without loss of generality that $X_1, \cdots, X_{2p} \in \mathscr{E}^{2n}$, from which

$$\begin{aligned} (\eta \wedge (d\eta)^p)(\xi, X_1, \cdots, X_{2p}) &= \eta(\xi)((d\eta)^p(X_1, \cdots, X_{2p})) \\ &= (d\eta)^p(X_1, \cdots, X_{2p}) \neq 0 \end{aligned}$$

where we have used the facts that $\eta(\xi) = 1$ and $\eta(X_1) = \cdots = \eta(X_{2p}) = 0$ for $X_1, \cdots, X_{2p} \in \mathscr{E}^{2n}$.

We now give some examples of quasi-Sasakian structures of odd rank. In fact we shall exhibit a large class of quasi-Sasakian manifolds.

Let M^{2n} be a Kaehler manifold with metric g'. Let Ω be the fundamental 2-form and J be the almost complex structure tensor. S. Kobayashi [6] has shown that the set of all principal circle bundles over M^{2n} can be given a group structure isomorphic to the cohomology group $H^2(M^{2n}, Z)$, where Z is the ring of integers. Using this result we can prove the following theorem.

Theorem 2.3. *Let M^{2n} be a Kaehler manifold. If there exists a 2-form Ψ^* of bidegree* (1, 1) *and rank p representing an element of $H^2(M^{2n}, Z)$, then there exists a quasi-Sasakian structure of rank $2p + 1$ on the corresponding principal circle bundle.*

Proof. Let M^{2n+1} denote the bundle space and $\pi : M^{2n+1} \to M^{2n}$ the projection map. Let η' be a connexion form on M^{2n+1}. Then there exists a 2-form $\Psi^{*\prime}$ on M^{2n} such that $d\eta' = \pi^*\Psi^{*\prime}$. However, the characteristic class $[\Psi^*]$ of M^{2n+1}, $[\Psi^*] \in H^2(M^{2n}, Z)$, is independent of the choice of connexions (Kobayashi [6]), so that $[\Psi^*] = [\Psi^{*\prime}]$. Thus, there exists a 1-form ω on M^{2n} such that $\Psi^* - \Psi^{*\prime} = d\omega$. Hence

$$\pi^*\Psi^* = \pi^*\Psi^{*\prime} + \pi^*d\omega = d(\eta' + \pi^*\omega) .$$

Now $\pi^*\omega$ is horizontal and *ad*-equivariant (i.e. $\pi^*\omega \circ R_s = ad(s^{-1})\pi^*\omega$, where R_s is right translation by $s \in S^1$). Since S^1 is abelian, $ad(s^{-1})$ is the identity map, so $\pi^*\omega \circ R_s = \pi^*\omega$. Hence, if we set $\eta = \eta' + \pi^*\omega$,

$$\eta \circ R_s = \eta ,$$

since η' is *ad*-equivariant. Moreover, if ξ is a vertical vector field such that $\eta'(\xi) = 1$, then $\eta(\xi) = 1$, since $(\pi^*\omega)(\xi) = \omega(\pi_*\xi) = 0$. Thus, η is a connexion form on M^{2n+1} with $d\eta = \pi^*\Psi^*$, the curvature form of η, and hence $\eta \wedge (d\eta)^p \neq 0$. For, if $X_1, \cdots, X_{2p}$ are $2p$ linearly independent horizontal vector fields,

$$\begin{aligned}(\eta \wedge (d\eta)^p)(\xi, X_1, \cdots, X_{2p}) &= \eta(\xi)((d\eta)^p(X_1, \cdots, X_{2p})) \\ &= (\pi^*\Psi^*)^p(X_1, \cdots, X_{2p}) \\ &= \Psi^{*p}(\pi_*X_1, \cdots, \pi_*X_{2p}) \neq 0 .\end{aligned}$$

Define ϕ by $\phi X = \tilde{\pi}J\pi_*X$ where $\tilde{\pi}$ denotes the horizontal lift with respect to the connexion η. Then, since ξ is vertical, $\phi\xi = 0$; moreover $\eta \circ \phi = 0$. An easy computation gives $\phi^2X = -X + \eta(X)\xi$. Hence, we have an almost contact structure on M^{2n+1}. Now define a metric g on M^{2n+1} by $g(X, Y) = g'(\pi_*X, \pi_*Y) + \eta(X)\eta(Y)$. Then, since g' is hermitian, one can verify that g satisfies equations (1.2), so we have an almost contact metric structure on

M^{2n+1}. Defining the fundamental 2-form Φ by $\Phi(X, Y) = g(X, \phi Y)$ we see that

$$\begin{aligned}\Phi(X, Y) &= g'(\pi_* X, \pi_* \phi Y) + \eta(X)\eta(\phi Y) \\ &= g'(\pi_* X, J\pi_* Y) = \Omega(\pi_* X, \pi_* Y)\end{aligned}$$

so that $\Phi = \pi^*\Omega$, and $d\Phi = 0$ since M^{2n} is Kaehlerian. Finally, since Ψ^* is of bidegree (1, 1) and M^{2n} is Kaehlerian, it follows from Proposition 1.3 that the almost contact metric structure is normal.

That quasi-Sasakian manifolds actually exist may be seen by taking M^{2n} to be the Kaehlerian product of Kaehler manifolds M^{2p} and $M^{2q}(p + q = n)$ and letting Ψ^* denote the fundamental 2-form of M^{2p} extended to be a form on M^{2n} vanishing over M^{2q}.

We shall now show that if ξ is a regular closed vector field on a quasi-Sasakian manifold M^{2n+1}, then M^{2n+1} has a circle bundle structure as in Theorem 2.3.

Theorem 2.4. *If M^{2n+1} has a quasi-Sasakian structure (ϕ, ξ, η, g) of rank $2p + 1$ with ξ a regular closed vector field, then M^{2n+1} has a principal circle bundle structure over a Kaehler manifold, the characteristic class of M^{2n+1} being $[\Psi^*]$ where $d\eta = \pi^*\Psi^*$; Ψ^* is of bidegree (1, 1) and rank p.*

Proof. By Proposition 1.2, M^{2n+1} has a circle bundle structure over a complex manifold M^{2n}. Let $\pi : M^{2n+1} \to M^{2n}$ be the bundle projection map and $\tilde{\pi}X$ the horizontal lift of a vector field X on M^{2n} with respect to the connexion given by η. The almost complex structure tensor J on M^{2n} is given by $JX = \pi_*\phi\tilde{\pi}X$ and J is well-defined as we saw in § 1. A direct computation gives $J^2 = -I$. Indeed,

$$\begin{aligned}J^2X &= \pi_*\phi\tilde{\pi}\pi_*\phi\tilde{\pi}X \\ &= \pi_*\phi(\phi\tilde{\pi}X - \eta(\phi\tilde{\pi}X)\xi) \\ &= \pi_*(-\tilde{\pi}X + \eta(\tilde{\pi}X)\xi) = -X .\end{aligned}$$

Now define a metric g' on M^{2n} by $g'(X, Y) = g(\tilde{\pi}X, \tilde{\pi}Y) - \eta(\tilde{\pi}X)\eta(\tilde{\pi}Y)$, and a 2-form Ω on M^{2n} by $\Omega(X, Y) = g'(X, JY)$. Then

$$g'(JX, JY) = g'(X, Y)\,, \;\; \Omega(X, Y) = \Phi(\tilde{\pi}X, \tilde{\pi}Y)$$

where Φ is the fundamental 2-form of the structure (ϕ, ξ, η, g). Thus, g' is hermitian and $\pi^*\Omega = \Phi$. Now Φ has rank n, and hence Ω does also. Furthermore $0 = d\Phi = d\pi^*\Omega = \pi^*(d\Omega)$ implies $d\Omega = 0$ since π^* is injective. Thus, M^{2n} is Kaehlerian. Finally there exists a 2-form Ψ^* on M^{2n} (the curvature form of η) such that $d\eta = \pi^*\Psi^*$ which by Proposition 1.3 is of bidegree (1, 1). Moreover, the characteristic class of M^{2n+1} is $[\Psi^*] \in H^2(M^{2n}, Z)$.

Remark. Combining the above results with the well-known Boothby-Wang fibration [3] it is seen that if M^{2n+1} has a quasi-Sasakian structure of rank $2p+1$ with ξ a regular closed vector field, and $[\Omega] \in H^2(M^{2n}, Z)$, then it also has a Sasakian structure.

3. Locally product quasi-Sasakian manifolds

Before proceeding with the results of this section, we require some new notation and to possibly alter the quasi-Sasakian structure to a more canonical form. Let (ϕ, ξ, η, g') be a quasi-Sasakian structure of rank $2p+1$ on a manifold M^{2n+1}. Let $\mathscr{E}^{2p}$ denote the submodule of $\mathscr{E}^{2n+1}$ on which $(d\eta)^p \neq 0$ and $\phi\mathscr{E}^{2p} = \mathscr{E}^{2p}$; since $d\eta(X, \xi) = 0$ by Lemma 2.1, $\mathscr{E}^{2p}$ is a submodule of $\mathscr{E}^{2n}$. Let $\mathscr{E}^{2q}$ denote the orthogonal complement of $\mathscr{E}^{2p}$ in $\mathscr{E}^{2n}$ and define maps ψ and θ by

$$\psi = \begin{cases} \phi|_{\mathscr{E}^{2p}} & \text{on } \mathscr{E}^{2p} \\ 0 & \text{on } \mathscr{E}^{2q} \\ 0 & \text{on } \mathscr{E}^{1} \end{cases}, \qquad \theta = \begin{cases} 0 & \text{on } \mathscr{E}^{2p} \\ \phi|_{\mathscr{E}^{2q}} & \text{on } \mathscr{E}^{2q} \\ 0 & \text{on } \mathscr{E}^{1} \end{cases}$$

then $\phi = \psi + \theta$. Observe that ψ and hence θ are not unique since the choice of $\mathscr{E}^{2p}$ is not so. Now, if necessary, define a new metric g on M^{2n+1} by requiring that g agree with g' on $\mathscr{E}^{2q}$, and $\mathscr{E}^{1}$ and satisfy $g(X, \psi Y) = d\eta(X, Y)$ for $X, Y \in \mathscr{E}^{2p}$. It is easy to verify that (ϕ, ξ, η, g) is a quasi-Sasakian structure of rank $2p+1$ on M^{2n+1}. We shall work with this structure in this paper.

The maps $-\psi^2 + \xi \otimes \eta$ and $-\theta^2$ define an almost product structure with decomposition $\mathscr{E}^{2n+1} = \mathscr{E}^{2p+1} \oplus \mathscr{E}^{2q}$, where $\mathscr{E}^{2p+1} = \mathscr{E}^{2p} \oplus \mathscr{E}^{1}$. Similarly the maps $-\psi^2$ and $-\theta^2 + \xi \otimes \eta$ give an almost product structure with decomposition $\mathscr{E}^{2n+1} = \mathscr{E}^{2p} \oplus \mathscr{E}^{2q+1}$, where $\mathscr{E}^{2q+1} = \mathscr{E}^{2q} \oplus \mathscr{E}^{1}$. The integrability of these almost product structures is discussed below in detail.

Theorem 3.1. *If M^{2n+1} has a quasi-Sasakian structure of rank $2p+1$ with $[\theta, \theta] = 0$ for some θ, then M^{2n+1} is locally the product of a Sasakian manifold M^{2p+1} and a Kaehler manifold M^{2q}, $q = n - p$.*

Proof. It is well-known that $[\theta, \theta] = 0$ if and only if $[-\theta^2, -\theta^2] = 0$; but this is just the integrability condition for a locally product structure (the decomposition here is $\mathscr{E}^{2n+1} = \mathscr{E}^{2p+1} \oplus \mathscr{E}^{2q}$). Let x^a $(a = 1, \cdots, 2p+1)$, x^α $(\alpha = 2p+2, \cdots, 2n+1)$ be coordinates such that $\{\partial/\partial x^a\}$ is a basis of $\mathscr{E}^{2p+1}$, and $\{\partial/\partial x^\alpha\}$ is a basis of $\mathscr{E}^{2q}$. Thus if $\{x^a, x^\alpha\}$ and $\{y^a, y^\alpha\}$ are coordinates for two over lapping coordinate neighborhoods, then

$$\begin{vmatrix} \dfrac{\partial y^a}{\partial x^a} & \dfrac{\partial y^a}{\partial x^\alpha} \\ \dfrac{\partial y^\alpha}{\partial x^a} & \dfrac{\partial y^\alpha}{\partial x^\alpha} \end{vmatrix} \neq 0 .$$

However, since we have a locally product structure, the y^a's depend only on the x^a's, and the y^α's only on the x^α's; hence

$$\frac{\partial y^a}{\partial x^\alpha} = 0\,, \qquad \frac{\partial y^\alpha}{\partial x^a} = 0\,.$$

Therefore,

$$\left|\frac{\partial y^a}{\partial x^a}\right| \neq 0\,, \qquad \left|\frac{\partial y^\alpha}{\partial x^\alpha}\right| \neq 0\,.$$

Hence the system of subspaces defined by $x^\alpha =$ constant, for each α, is an atlas determining a manifold M^{2p+1}; similarly, the system of subspaces defined by $x^a =$ constant, for each a, is an atlas determining a manifold M^{2q}. Locally, M^{2n+1} is the product of M^{2p+1} and M^{2q}, and the localized modules of vector fields over M^{2p+1} and M^{2q} are (isomorphic to) $\mathscr{E}^{2p+1}$ and $\mathscr{E}^{2q}$, respectively.

Now $\eta \wedge (d\eta)^p \neq 0$ on $\mathscr{E}^{2p+1} = \mathscr{E}^{2p} \oplus \mathscr{E}^1$, so $\eta|_{\mathscr{E}^{2p+1}} \wedge (d(\eta|_{\mathscr{E}^{2p+1}}))^p \neq 0$ over M^{2p+1} giving a contact structure. Since ψ and ϕ agree on $\mathscr{E}^{2p}$ and vanish on $\mathscr{E}^1$, $(\psi, \xi, \eta)|_{\mathscr{E}^{2p+1}}$ satisfy equations (1.1) on M^{2p+1}. Furthermore, $g|_{\mathscr{E}^{2p+1}}$ satisfies equations (1.2). Hence, $(\psi, \xi, \eta, g)|_{\mathscr{E}^{2p+1}}$ is an associated almost contact metric structure. To show that M^{2p+1} is Sasakian, it remains only to show that the structure is normal. For $X, Y \in \mathscr{E}^{2p+1}$

$$\begin{aligned}
&[\psi, \psi](X, Y) + d(\eta|_{\mathscr{E}^{2p+1}})(X, Y)\xi \\
&\quad = [\phi, \phi](X, Y) - 2[\phi, \theta](X, Y) + [\theta, \theta](X, Y) + d\eta(X, Y)\xi \\
&\quad = -\, 2[\phi, \theta](X, Y)
\end{aligned}$$

since $\psi = \phi - \theta$, $[\theta, \theta] = 0$, and by normality $[\phi, \phi](X, Y) + d\eta(X, Y)\xi = 0$. Continuing the computation we have

$$\begin{aligned}
[\psi, \psi](X, Y) + d(\eta|_{\mathscr{E}^{2p+1}})(X, Y)\xi &= -\, 2[\phi, \theta](X, Y) \\
&= -\, (\phi\theta[X, Y] + \theta\phi[X, Y] + [\phi X, \theta Y] + [\theta X, \phi Y] \\
&\qquad - \phi[X, \theta Y] - \theta[X, \phi Y] - \phi[\theta X, Y] - \theta[\phi X, Y]) \\
&= 0
\end{aligned}$$

where each term in the last expression vanishes because $X, Y \in \mathscr{E}^{2p+1}$ and θ is zero on $\mathscr{E}^{2p+1}$, $X \in \mathscr{E}^{2p+1} = \mathscr{E}^{2p} \oplus \mathscr{E}^1$ implies $\phi X = \psi X + \theta X \in \mathscr{E}^{2p}$, and the distributions $-\psi^2 + \xi \otimes \eta$ and $-\theta^2$ are integrable so that $[X, Y] \in \mathscr{E}^{2p+1}$.

Finally, define a 2-form Θ by $\Theta(X, Y) = g(X, \theta Y)$. Since $\theta = \phi - \psi$ we have $\Theta = \Phi - \Psi$, and hence $d\Theta = 0$. Furthermore, θ has rank $2q$, so $\Theta^q \neq 0$, $(\theta|_{\mathscr{E}^{2q}})^2 = -I$, $[\theta, \theta] = 0$ and $g|_{\mathscr{E}^{2q}}$ is hermitian. Thus, $\theta|_{\mathscr{E}^{2q}}$ and $g|_{\mathscr{E}^{2q}}$ give M^{2q} a Kaehler structure.

We can also obtain the converse of this theorem, so we again have a large class of examples of quasi-Sasakian manifolds.

Theorem 3.2. *If a manifold M^{2n+1} is (locally) the product of a Sasakian manifold M^{2p+1} and a Kaehler manifold M^{2q}, then M^{2n+1} has a quasi-Sasakian structure of rank $2p+1$.*

Proof. Let $\mathscr{E}^{2n+1}$, $\mathscr{E}^{2p+1}$ and $\mathscr{E}^{2q}$ denote the localized modules of vector fields on M^{2n+1}, M^{2p+1} and M^{2q} respectively. Then, since M^{2n+1} is locally the product of M^{2p+1} and M^{2q}, we have $\mathscr{E}^{2n+1} \cong \mathscr{E}^{2p+1} \oplus \mathscr{E}^{2q}$. Let $(\phi', \xi', \eta', g_p)$ be an associated almost contact metric structure to the Sasakian structure on M^{2p+1}, and (θ', g_q) an associated almost hermitian structure on M^{2q} (i.e. $\theta'^2 = -I$, $g_q(\theta' X, \theta' Y) = g_q(X, Y)$, $X, Y \in \mathscr{E}^{2q}$, and $\Theta'(X, Y) = g_q(X, \theta' Y)$ where Θ' is the fundamental 2-form of the Kaehler structure on M^{2q}).

We shall write $X \in \mathscr{E}^{2n+1}$ as $X_1 + X_2$ where $X_1 \in \mathscr{E}^{2p+1}$ and $X_2 \in \mathscr{E}^{2q}$ (under the isomorphism $\mathscr{E}^{2n+1} \cong \mathscr{E}^{2p+1} \oplus \mathscr{E}^{2q}$). Define a 1-form η on M^{2n+1} by $\eta(X) = \eta'(X_1)$ and take $\xi \in \mathscr{E}^{2n+1}$ to be equal to $\xi' \in \mathscr{E}^{2p+1}$; then $\eta(\xi) = \eta'(\xi') = 1$. Now define new maps $\psi, \theta, \phi : \mathscr{E}^{2n+1} \to \mathscr{E}^{2n+1}$ by $\psi X = \phi' X_1$, $\theta X = \theta' X_2$, $\phi = \psi + \theta$; then a direct verification gives $\phi\xi = 0$, $\eta \circ \phi = 0$ and $\phi^2 = -I + \xi \otimes \eta$. Defining a metric g on M^{2n+1} by $g = g_p + g_q$, we obtain equations (1.2) by direct computation using the facts that g_p satisfies equations (1.2) and g_q is hermitian. Thus, M^{2n+1} has an almost contact metric structure.

Now since M^{2n+1} has a locally product structure, we have coordinates $\{x^a, x^\alpha\}$ and basis vector fields $\{\partial/\partial x^a, \partial/\partial x^\alpha\}$ as in the proof of Theorem 3.1. With respect to this basis the components ψ_a^b of ψ are functions of the x^a's alone ($\psi_a^b \partial/\partial x^b = \psi \partial/\partial x^a = \phi' \partial/\partial x^a$) and similarly for the components θ_α^β of θ. Using these facts a direct computation yields $[\phi, \phi] + \xi \otimes d\eta = 0$ giving the normality of the structure on M^{2n+1}.

Finally let Φ, given by $\Phi(X, Y) = g(X, \phi Y)$, denote the fundamental 2-form of the structure on M^{2n+1}. Since M^{2q} is Kaehlerian, $d\Theta' = 0$, and since $\phi = \psi + \theta\,(= \phi' + \theta')$, $\Phi = d\eta' + \Theta'$. Thus $d\Phi = d(d\eta' + \Theta') = 0$. Hence, the almost contact metric structure defined above is quasi-Sasakian; since ϕ' has rank $2p$, so has ψ and therefore the structure has rank $2p+1$.

It should be remarked that quasi-Sasakian structures with $[\psi, \psi] = 0$ (decomposition $\mathscr{E}^{2n+1} = \mathscr{E}^{2p} \oplus \mathscr{E}^{2q+1}$, where $\mathscr{E}^{2q+1} = \mathscr{E}^{2q} \oplus \mathscr{E}^1$) are of special interest. In fact, if $[\psi, \psi] = 0$ then ψ is the zero map and we therefore have only the rank 1 case (cosymplectic). For, since $[\psi, \psi] = 0$ gives an integrable distribution, $X, Y \in \mathscr{E}^{2p}$ implies $[X, Y] \in \mathscr{E}^{2p}$, and hence $[\theta, \theta](X, Y) = 0$. Now, by normality

$$\begin{aligned} -d\eta(X, Y)\xi &= [\phi, \phi](X, Y) \\ &= [\psi, \psi](X, Y) + 2[\psi, \theta](X, Y) + [\theta, \theta](X, Y) \\ &= 2[\psi, \theta](X, Y) = 0\,. \end{aligned}$$

But if X or Y is in $\mathscr{E}^{2q+1}$ then $d\eta(X, Y) = g(X, \psi Y) = 0$. Thus, $d\eta(X, Y) = 0$ for every $X, Y \in \mathscr{E}^{2n+1}$ giving the cosymplectic case.

The integrability of the almost product structure determined by $-\phi^2$, and $\xi \otimes \eta$ (decomposition $\mathscr{E}^{2n+1} = \mathscr{E}^{2n} \oplus \mathscr{E}^1$) also occurs only in the cosymplectic case. For, we know that $[-\phi^2, -\phi^2] = 0$ ($[\xi \otimes \eta, \xi \otimes \eta] = 0$) if and only if $[\phi, \phi] = 0$, and hence it follows from the normality condition,

$$[\phi, \phi](X, Y) + d\eta(X, Y)\xi = 0 ,$$

that $[\phi, \phi] = 0$ if and only if $d\eta = 0$.

Returning to the case $[\theta, \theta] = 0$, let us suppose that ξ is a regular closed vector field and see what effect the integrability condition has on the base space of Theorems 2.3 and 2.4.

Theorem 3.3. *Let M^{2n+1} be a quasi-Sasakian manifold of rank $2p + 1$ with ξ a regular closed vector field. If M^{2n+1} has the locally product structure of Theorem 3.1* (i.e. $[\theta, \theta] = 0$), *then the base manifold M^{2n} of the circle bundle M^{2n+1} is locally the Kaehlerian product of two Kaehler manifolds M^{2p} and M^{2q}, $p + q = n$.*

Proof. Using the usual notation, define maps P and Q over M^{2n} by $PX = \pi_*\psi\tilde{\pi}X$ and $QX = \pi_*\theta\tilde{\pi}X$ where $\tilde{\pi}X$ is the horizontal lift of X with respect to the connexion η on M^{2n+1}. Then, since $\phi = \psi + \theta$, the almost complex structure tensor J on M^{2n} satisfies $J = P + Q$. Since $\psi\theta = \theta\psi = 0$, it can be verified that $PQ = QP = 0$ and $-P^2 + (-Q^2) = I$, and hence $-P^2$ and $-Q^2$ are projection maps. Thus, to show that M^{2n} has a locally product structure one needs only to show that $[Q, Q] = 0$, for, then $[-Q^2, -Q^2] = [-P^2, -P^2] = 0$. Thus, if X and Y are vector fields on M^{2n}

$$\begin{aligned}
[Q, Q](X, Y) &= Q^2[X, Y] + [QX, QY] - Q[X, QY] - Q[QX, Y] \\
&= \pi_*\theta^2\tilde{\pi}[X, Y] + [\pi_*\theta\tilde{\pi}X, \pi_*\theta\tilde{\pi}Y] - \pi_*\theta\tilde{\pi}[X, \pi_*\theta\tilde{\pi}Y] \\
&\quad - \pi_*\theta\tilde{\pi}[\pi_*\theta\tilde{\pi}X, Y] \\
&= \pi_*\theta^2[\tilde{\pi}X, \tilde{\pi}Y] + \pi_*[\theta\tilde{\pi}X, \theta\tilde{\pi}Y] - \pi_*\theta[\tilde{\pi}X, \theta\tilde{\pi}Y] \\
&\quad - \pi_*\theta[\theta\tilde{\pi}X, \tilde{\pi}Y] \\
&= 0 .
\end{aligned}$$

The spaces of which M^{2n} is locally the product will be denoted by M^{2p} and M^{2q}; we now show that these spaces are Kaehlerian. Since $-P^2$ and $-Q^2$ are projection maps we have $P^2|_{\mathscr{E}^{2p}} = -I|_{\mathscr{E}^{2p}}$ and $Q^2|_{\mathscr{E}^{2q}} = -I|_{\mathscr{E}^{2q}}$ giving almost complex structures on M^{2p} and M^{2q}; furthermore, we have $[P, P] = 0$ and $[Q, Q] = 0$ so these are complex structures. If g' is the Kaehler metric on M^{2n}, then for $X, Y \in \mathscr{E}^{2p}$

$$g'|_{\mathscr{E}^{2p}}(PX, PY) = g'(JX, JY) = g'(X, Y) = g'|_{\mathscr{E}^{2p}}(X, Y) .$$

Similarly $g'|_{\mathscr{E}^{2q}}(QX, QY) = g'|_{\mathscr{E}^{2q}}(X, Y)$ for $X, Y \in \mathscr{E}^{2q}$. Thus, the restrictions of g' to $\mathscr{E}^{2p}$ and $\mathscr{E}^{2q}$ give hermitian metrics on M^{2p} and M^{2q}, respectively. Define 2-forms Ω_1 and Ω_2 by $\Omega_1(X, Y) = g'(X, PY)$ and $\Omega_2(X, Y) = g'(X, QY)$. Then, since $J = P + Q$, the fundamental 2-form Ω on M^{2n} is equal to $\Omega_1 + \Omega_2$. Since P has rank $2p$ and Q rank $2q$, $\Omega_1^p \neq 0$ on M^{2p} and $\Omega_2^q \neq 0$ on M^{2q}. Finally, since $\Omega_1|_{\mathscr{E}^{2p}} = \Omega|_{\mathscr{E}^{2p}}$ and $\Omega_2|_{\mathscr{E}^{2q}} = \Omega|_{\mathscr{E}^{2q}}$, $d(\Omega_1|_{\mathscr{E}^{2p}}) = 0$ and $d(\Omega_2|_{\mathscr{E}^{2q}}) = 0$.

Now $\nabla_X\theta = 0$ for every X implies $[\theta, \theta] = 0$ where ∇ is covariant differentiation with respect to the Riemannian connexion determined by the metric g of the quasi-Sasakian structure. Thus, if the stronger hypothesis is imposed we have a locally product structure as above. We show that it is actually a *locally decomposable* Riemannian structure, i.e., if $\{x^a, x^\alpha\}$ are the coordinates introduced above, then $g(\partial/\partial x^a, \partial/\partial x^b)$, $a, b = 1, \cdots, 2p+1$, depends only on the x^a's, and $g(\partial/\partial x^\alpha, \partial/\partial x^\beta)$, $\alpha, \beta = 2p+2, \cdots, 2n+1$, only on the x^α's.

Lemma 3.4. *$\nabla_X\theta = 0$ implies $\nabla_X\theta^2 = 0$.*

Proof. $(\nabla_X\theta)Y = \nabla_X\theta Y - \theta\nabla_X Y$.

Hence

$$\begin{aligned}(\nabla_X\theta^2)Y &= \nabla_X\theta^2 Y - \theta^2\nabla_X Y \\ &= \theta\nabla_X\theta Y + (\nabla_X\theta)\theta Y + \theta(\nabla_X\theta)Y - \theta\nabla_X\theta Y \\ &= (\nabla_X\theta)\theta Y + \theta(\nabla_X\theta)Y\end{aligned}$$

from which the lemma follows.

Let $F = -\phi^2 + \xi\otimes\eta + \theta^2$, that is, F is the difference of the projection maps $-\phi^2 + \xi\otimes\eta$ and $-\theta^2$. It is known (Yano [12], p. 221) that a necessary and sufficient condition for a locally product Riemannian space to be locally decomposable is

$$\nabla_X F = 0$$

for every X. But in our case $F = -\phi^2 + \xi\otimes\eta + \theta^2 = I + 2\theta^2$ and our result follows from the lemma. We state the result formally.

Theorem 3.5. *If M^{2n+1} has a quasi-Sasakian structure of rank $2p+1$ with $\nabla_X\theta = 0$ for every $X \in \mathscr{E}^{2n+1}$, then M^{2n+1} is a locally decomposable Riemannian manifold with the locally product structure of Theorem 3.1.*

Corollary 3.6. *The Riemannian structure of a cosymplectic manifold is locally decomposable.*

4. Some geometric results

In the last section we considered the distributions $-\theta^2$ and $-\phi^2 + \xi\otimes\eta$ with $[\theta, \theta] = 0$; here we shall work with a general quasi-Sasakian structure, that is, we have the three distributions $-\phi^2$, $-\theta^2$, $\xi\otimes\eta$. We begin with some important and interesting lemmas.

Lemma 4.1. *The fundamental vector field ξ of a quasi-Sasakian structure is a Killing vector field.*

Proof. By normality $N^{(3)}$ vanishes, that is, $\mathscr{L}_\xi\phi = 0$; hence for $X, Y \in \mathscr{E}^{2n+1}$

$$(\mathscr{L}_\xi\Phi)(X, Y) = (\mathscr{L}_\xi g)(X, \phi Y) .$$

On the other hand

$$\mathscr{L}_\xi\Phi = d\iota_\xi\Phi + \iota_\xi d\Phi = 0$$

since $d\Phi = 0$ and $(\iota_\xi\Phi)X = \Phi(\xi, X) = g(\xi, \phi X) = 0$. Furthermore, by normality, $N^{(4)}$ vanishes, that is, $\mathscr{L}_\xi\eta = 0$. Hence, a computation gives

$$(\mathscr{L}_\xi g)(X, (\xi\otimes\eta)Y) = 0 .$$

Thus $(\mathscr{L}_\xi g)(X, \phi Y) = 0$ and $(\mathscr{L}_\xi g)(X, (\xi\otimes\eta)Y) = 0$. Now since the map $\phi + \xi\otimes\eta$ is non-singular, $\mathscr{L}_\xi g = 0$ and hence ξ is a Killing vector field.

Lemma 4.2. *$\mathscr{L}_\xi\psi = 0$ and $\mathscr{L}_\xi\theta = 0$.*

Proof. $\mathscr{L}_\xi\Psi = 0$, since $d\Psi = 0$ and $(\iota_\xi\Psi)X = 0$. Hence

$$0 = (\mathscr{L}_\xi\Psi)(X, Y) = \xi(g(X, \psi Y)) - g([\xi, X], \psi Y) - g(X, \psi[\xi, Y]) .$$

However, by Lemma 4.1

$$0 = (\mathscr{L}_\xi g)(X, \psi Y) = \xi(g(X, \psi Y)) - g([\xi, X], \psi Y) - g(X, [\xi, \psi Y]) .$$

Therefore, $0 = [\xi, \psi Y] - \psi[\xi, Y] = (\mathscr{L}_\xi\psi)Y$. On the other hand, since $\mathscr{L}_\xi\phi = 0$, we have $\mathscr{L}_\xi\theta = \mathscr{L}_\xi\phi - \mathscr{L}_\xi\psi = 0$.

Lemma 4.3. *$\nabla_Y\xi = -\dfrac{1}{2}\psi Y$ for any $Y \in \mathscr{E}^{2n+1}$ (∇ is covariant differentiation with respect to the Riemannian connexion).*

Proof. Since ∇ is covariant differentiation with respect to the Riemannian connexion

$$(\nabla_X\eta)(Y) - (\nabla_Y\eta)(X) = d\eta(X, Y) .$$

Using the fact that ξ is a Killing vector field, that is,

$$0 = \xi g(X, Y) - g([\xi, X], Y) - g(X, [\xi, Y]) ,$$

and the identity

$$Xg(Y, Z) = g(Y, \nabla_X Z) + g(\nabla_X Y, Z)$$

we obtain

$$(\nabla_X\eta)(Y) = -(\nabla_Y\eta)(X) .$$

Hence,

$$d\eta(X, Y) = -2(\nabla_Y\eta)(X) ,$$

so $g(X, \phi Y) = -2g(X, \nabla_Y\xi)$ from which since X is arbitrary

$$\nabla_Y\xi = -\frac{1}{2}\phi Y$$

as desired.

The questions of the distributions $-\phi^2$, $-\theta^2$, $\xi \otimes \eta$ being parallel along one another, being flat and being geodesic were discussed in detail in [1]. Here we will prove an interesting curvature theorem.

Let X_m, Y_m be tangent vectors at $m \in M^{2n+1}$ and $K(X_m, Y_m)$ denote the sectional curvature at m determined by the plane section spanned by X_m and Y_m. Let R_{XY} denote the curvature transformation, that is,

$$R_{XY} = \nabla_{[X,Y]} + \nabla_Y\nabla_X - \nabla_X\nabla_Y .$$

Theorem 4.4. *If M^{2n+1} has a quasi-Sasakian structure of rank $2p+1$ then at every $m \in M^{2n+1}$*

$$K(\xi_m, X_m) = \begin{cases} \frac{1}{4}, & X_m \in \mathscr{E}^{2p+1}(m),\ X_m \notin \mathscr{E}^1(m) \\ 0, & X_m \in \mathscr{E}^{2q}(m) . \end{cases}$$

Proof. Without loss of generality we may take X_m to be a unit vector orthogonal to ξ_m. We now have

$$\begin{aligned} g(R_{\xi X}\xi, X) &= g(\nabla_{[\xi,X]}\xi, X) + g(\nabla_X\nabla_\xi\xi, X) - g(\nabla_\xi\nabla_X\xi, X) \\ &= g(-\frac{1}{2}\phi[\xi, X], X) - g(-\frac{1}{2}\nabla_\xi\phi X, X) \\ &= g(-\frac{1}{2}\phi[\xi, X], X) - g(-\frac{1}{2}\nabla_{\phi X}\xi, X) - g(-\frac{1}{2}[\xi, \phi X], X) \\ &= -g(\frac{1}{4}\phi^2 X, X) \\ &= \begin{cases} \frac{1}{4}, & X \in \mathscr{E}^{2p} \\ 0, & X \in \mathscr{E}^{2q} \end{cases} \end{aligned}$$

from which the result follows. We have used Lemmas 4.2 and 4.3 in the computation.

In the Sasakian case (rank $2n+1$) the theorem reduces to that of Hatakeyama, Ogawa, Tanno [4].

Corollary 4.5. *A quasi-Sasakian manifold of constant curvature is either Sasakian or cosymplectic (locally flat).*

Corollary 4.6. *A quasi-Sasakian manifold of strictly positive curvature is Sasakian.*

5. Characterization of the cosymplectic case

The following formula in the theory of Sasakian manifolds is proved in [11]:

$$(\nabla_X\Phi)(Y,Z) = \frac{1}{2}(\eta(Y)g(X,Z) - \eta(Z)g(X,Y)) . \tag{5.1}$$

In this section its quasi-Sasakian analogue is given in order to determine the meaning of the vanishing of $\nabla_X\Phi$ (equivalently $\nabla_X\phi$).

Proposition 5.1. *On a quasi-Sasakian manifold*

$$\begin{aligned}(\nabla_X\Phi)(Y,Z) &= \frac{1}{2}(\eta(Y)g(X,Z) - \eta(Z)g(X,Y)) \\ &\quad + \frac{1}{2}(\eta(Y)g(\theta^2X,Z) - \eta(Z)g(\theta^2X,Y)) .\end{aligned} \tag{5.2}$$

The proof is a very lengthy computation but is similar to that of (5.1).

Theorem 5.2. *A quasi-Sasakian manifold M^{2n+1} is cosymplectic (rank 1) if and only if $\nabla_X\Phi = 0$ for every $X \in \mathscr{E}^{2n+1}$.*

Proof. The condition is clearly sufficient; for, $\nabla_X\Phi = 0$ for every X, implies $[\phi,\phi] = 0$ and hence by normality we have

$$d\eta(X,Y)\xi = -[\phi,\phi](X,Y) = 0$$

for every $X, Y \in \mathscr{E}^{2n+1}$. Necessity follows from Proposition 5.1; for, if $d\eta = 0$ on M^{2n+1}, then ϕ is the zero map on $\mathscr{E}^{2p+1}$ and $\theta = \phi$. Thus (5.2) becomes

$$\begin{aligned}(\nabla_X\Phi)(Y,Z) &= \frac{1}{2}(\eta(Y)g(X,Z) - \eta(Z)g(X,Y)) \\ &\quad + \frac{1}{2}(\eta(Y)g(-X+\eta(X)\xi,Z) \\ &\quad - \eta(Z)g(-X+\eta(X)\xi,Y)) \\ &= 0 .\end{aligned}$$

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